ACCURACY OF THE MINIMAL CUT APPROXIMATION OF OF RELIABILITY FOR K-OUT-OF-N SYSTEMS

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THESIS

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ABSTRACT

The minimal cut lower bound for k-out-of-n systems is computed and compared with the true reliability of these systems. The size of the system, n, is increased; and selected degrees of system complexity, k/n, are studied. The resulting graphs of system reliability versus component reliability indicate that both size and complexity cause a deterioration of the approximation, but they also indicate that there is a limit to this deterioration. The minimal cut lower bound is then examined, theoretically, as the size of the system increases to infinity; and the limits of deterioration are obtained.

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I. INTRODUCTION

Reliability, the probability that a device will accomplish the mission for which it was designed, has become increasingly more difficult to compute as devices have become larger and more complex. It is often not feasible, even with the use of large computers, to calculate the actual reliability of relatively simple systems. When systems such as the Apollo mooncraft are considered, then clearly the task becomes formidable, if not impossible. Knowledge of the reliability of such systems, however, is vitally important.

To compensate for this inability to compute actual system reliability, certain methods for approximating reliability have been developed. Usually, these approximations tend to place lower or upper bounds on the actual reliability and to become arbitrarily close as the performance probability of the components increases to unity. In many cases, however, much analysis remains to be done to determine the strengths and weaknesses of these approximations.

While the reliability of a k-out-of-n system can be found directly and needs no approximation, the study of such systems will allow the characteristics of the minimal cut lower bound method of approximating reliability to be thoroughly analyzed. This paper will conduct such an analysis.

II. BACKGROUND AND DISCUSSION

A. DEFINITIONS

Reliability is usually thought of as being time dependent; that is, it is the probability that a device functions properly over the interval $[0,t]$. Implicit in the definition of reliability is the assumption that the system or device has two states: success or failure. Birnbaum, Esary, and Saunders [2] refer to this definition as "dichotomic reliability." Throughout this paper use of the term reliability will imply dichotomic reliability. This same assumption will also be used when referring to components and their performance probabilities.

A system is said to be in logical series if and only if all components of the system must perform in order that the system can function.

A system is in logical parallel if and only if at least one component must perform in order for the system to perform.

B. k-OUT-OF-n SYSTEMS

The k-out-of-n system was chosen because it is often found in practice, and the true reliability of such a system is easily computed or acquired from tables. This type of system functions when at least k of its n components perform properly and fails otherwise. Examples of such a system would include a suspension bridge which needs at least k of its n cables to remain standing; or a cable consisting of n wires, k of which are vital to support the maximum load. It is hard to conceive of such a system in which the n components would not be identical, and this assumption is usually made. The assumption is also

made that a component either performs properly or fails completely, and that this action is independent of all other components. Thus, the reliability of such systems, R_s, can be obtained by using the equation for the cumulative binomial probability distribution

$$R_s = \sum_{k'=k}^{n} \binom{n}{k'} p^{k'} (1-p)^{n-k'} , \qquad \text{(II-B-1)}$$

where p is the probability that the individual components perform properly. The reliability can also be obtained from numerous cumulative binomial tables covering a wide range of values for k, n, and p. If all components were not identical and independent, then equation (II-B-1) would not hold and all possible combinations of k components would have to be enumerated.

C. MINIMAL CUT LOWER BOUND

In every system of n components there is a group of components which, by performing, insures that the system performs. This group or set of components is called a "path" of the system. Depending on the redundancy built into the system, the total number of paths possible can vary widely; and each path can contain from 1 to n components. Within each path of the system there exists a minimal group of components whose performance is absolutely essential to the functioning of the system. This set of components is called a "minimal path set." Esary and Proschan [4], in a more formal definition, describe a minimal path set as a path of which no proper subset is also a path.

In a similar fashion, there are within a system certain sets, consisting of a minimal group of components, whose failure would cause the system to fail. Each of these sets would be called a "minimal cut

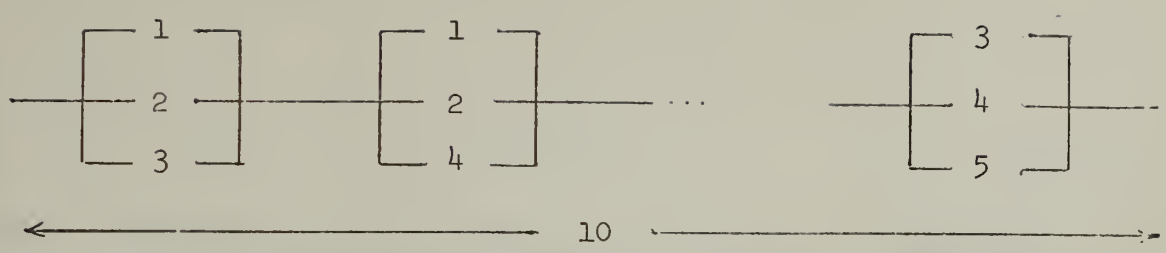
set." In order for one of these minimal cut sets to fail, all the components belonging to that set must fail. This means that the minimal cut set can be thought of as forming a parallel structure or a subsystem with all its components arranged in parallel. Since there can be more than one minimal cut set in a system, it is entirely possible, and even likely, that a particular component could appear in several of these sets. The failure of any one of the minimal cut sets is sufficient to fail the system; therefore, the collection of cut sets can be thought of as forming a series structure.

In the k-out-of-n systems, there are k components in a minimal path set, which means that (n-k) components could fail without affecting the performance of the system. Failure of one more, however, would cause the system to fail. Therefore, there are (n-k+1) components in a minimal cut set. The total number of minimal cut sets is, then, simply the number of subsets of size (n-k+1) that can be chosen from the set of n components. This quantity is given by:

$$\binom{n}{n-k+1} = \frac{n!}{(n-k+1)!\ (k-1)!} \quad . \qquad \text{(II-C-1)}$$

A physical representation of this structure would be $\binom{n}{n-k+1}$ cut set subsystems in series with each subsystem containing (n-k+1) components in parallel.

Example: In a 3-out-of-5 system, a minimal cut set would contain (n-k+1), or (5-3+1)=3, components in parallel; and there are $\binom{n}{n-k+1}$, or $\binom{5}{3} = \frac{5!}{3!\ 2!} = 10$, different ways to obtain these cut sets. The physical representation would appear as:



Theoretically, the representation with the minimal cut structures in series could be used to compute the reliability of the system. This would be simple and straight forward except when a single component appears in more than one minimal cut structure. Computing this reliability would then become a cumbersome task, to say the least. This issue can be avoided, however, by substituting identical and independent components in place of the repetitions. It is clear, however, that such substitutions would make the structure more likely to fail and would, in fact, form an upper bound on the probability that the structure fails, or a lower bound on the reliability of the system. This type of structure is then used to form the minimal cut lower bound on a system's reliability.[1]

In the k-out-of-n system it has been established that each minimal cut structure would have (n-k+1) components in parallel, and that $\binom{n}{n-k+1}$ of these structures would be connected in series. If p is the probability that an individual component functions properly, then (1-p) is the probability that it fails. The probability that all components of a minimal cut structure for a k-out-of-n system fail is:

$$(1-p)^{n-k+1} \, . \qquad \text{(II-C-2)}$$

[1]A detailed and mathematical proof of the validity of the minimal cut lower bound is given by J. D. Esary and F. Proschan in [3].

The probability that the cut structure does not fail, its reliability, is given by

$$1 - (1-p)^{n-k+1} \qquad \text{(II-C-3)}$$

Since there are $\binom{n}{n-k+1}$ cut structures in series, the approximate reliability of the system, R_a, using the minimal cut lower bound then becomes:

$$R_a = \left[1 - (1-p)^{n-k+1}\right]^{\binom{n}{n-k+1}} . \qquad \text{(II-C-4)}$$

Equation (II-C-4) now represents the basic equation for computing the minimal cut lower bound for any k-out-of-n system with identical and independent components.

For the logical series or logical parallel systems, the equation for the minimal cut lower bound reduces to an equation which computes the system's true reliability. This should be obvious, since each component is a minimal cut set in a logical series system; and in the logical parallel system, all the components form one minimal cut set. This fact can be seen in an elementary way by substituting into the two equations (II-B-1) and (II-C-4) the value k = n for the series system and k = 1 for the parallel system. The two equations would reduce to:

(a) series system (k = n) $R_s = R_a = p^n$, (II-C-5)

(b) parallel system (k = 1) $R_s = R_a = 1 - (1-p)^n$. (II-C-6)

The minimal cut lower bound is, therefore, ~~a perfect~~ approximation in the two extreme cases of the logical series or logical parallel systems; but how good is the approximation between these two limits?

III. NATURE OF THE ANALYSIS

A. THE PROBLEM

> The theoretical interest of the bounds is that they offer the means in reliability studies of approximating structures having complex component arrangements with structures having only series and parallel arrangements. The practical interest of the bounds is that they can be useful for structures having reliability functions tedious to evaluate exactly, but whose paths and cuts can be determined by inspection. Such structures are quite numerous.[1]

While the validity of the minimal cut lower bound has been proven, much remains to be done to determine just how "useful" it is in the practical sense. Is the minimal cut lower bound always a good approximation to a system's true reliability? How does the lower bound behave as the complexity or the number of components increases? Does the approximation depend in any way on the ratios of components in the cut sets to the total number of system components? If the minimal cut lower bound is not a good approximation, what causes the deterioration? Can the amount of deterioration be determined or defined?

The study of k-out-of-n systems will allow some light to be cast on these and related questions and will aid in developing a stronger base for theoretical implications.

B. ANALYSIS TECHNIQUE

The problem was investigated in the following manner: A base system of n=10 components was chosen, and k was allowed to vary from 1 to 10. The true system reliability for each value of k was obtained from

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[1]Barlow, R. E. and Proschan, F., Mathematical Theory of Reliability, p. 207, John Wiley and Sons, Inc., 1968.

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cumulative binomial tables, rounded off to four significant decimals, and plotted on a graph of system reliability versus component performance probability. (Figure III-C.1) The minimal cut lower bound for each k was then calculated using equation (II-C-4) on an IBM-360 computer. The resulting calculations, rounded off to four significant decimals, were plotted on a graph similar to the one above. (Figure III-C.2) Maintaining the ratios of k to n established in the base system, the value of n was increased to 20, 40, and 80; and the resulting true system reliability and minimal cut lower bound were obtained in the manner described above. (See Appendix A for values.) Due to the size of the numbers involved, 80 factorial, it was not feasible to increase the value of n past 80. This was not, however, a limiting factor in the analysis.

In order to study the results more closely, k to n ratios of .2, .5, and .8 were selected; and the values of both true reliability and the lower bound for these ratios were plotted for each system of size n. (See Figures III-C.3, C.4, C.5, C.6.) These four graphs were then used to answer questions raised in part A above and to establish any developing trends.

C. DISCUSSION OF GRAPHS

The graph of figure III-C.1 was plotted to show the relative "S-shapedness" of the k-out-of-n systems and to show the family of curves which results as k varies from the logical parallel system to the logical series system. The notion of S-shapedness is discussed in detail in references [2], [3], and [4].

True Reliability of the k-Out-of-10 Systems

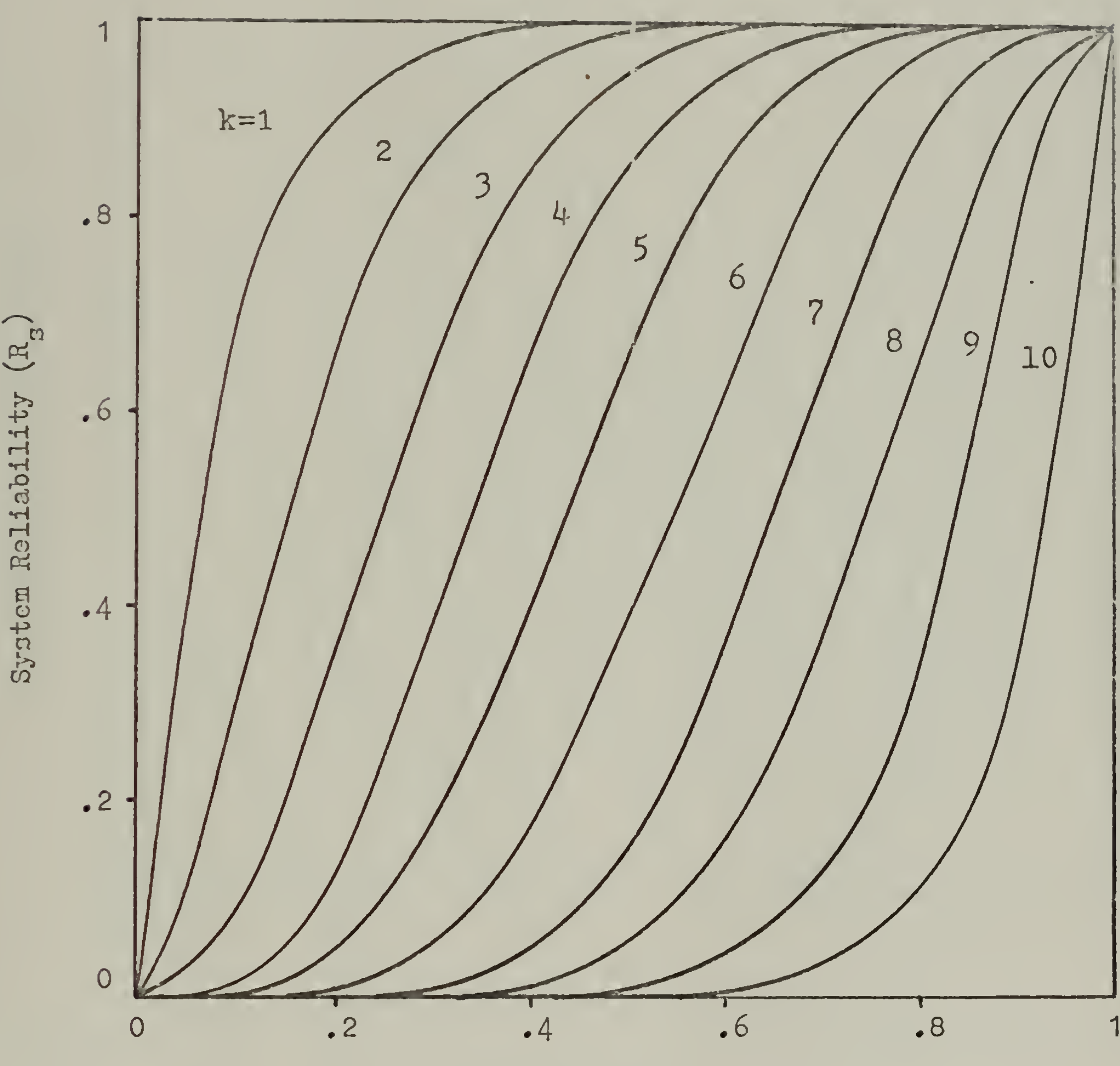

Figure III-C.1

The graph of figure III-C.2 shows the family of curves generated by the minimal cut lower bound for the k-out-of-10 systems. The graph shows clearly that the lower bound and the true system reliability are

Minimal Cut Lower Bound of the k-Out-of-10 Systems

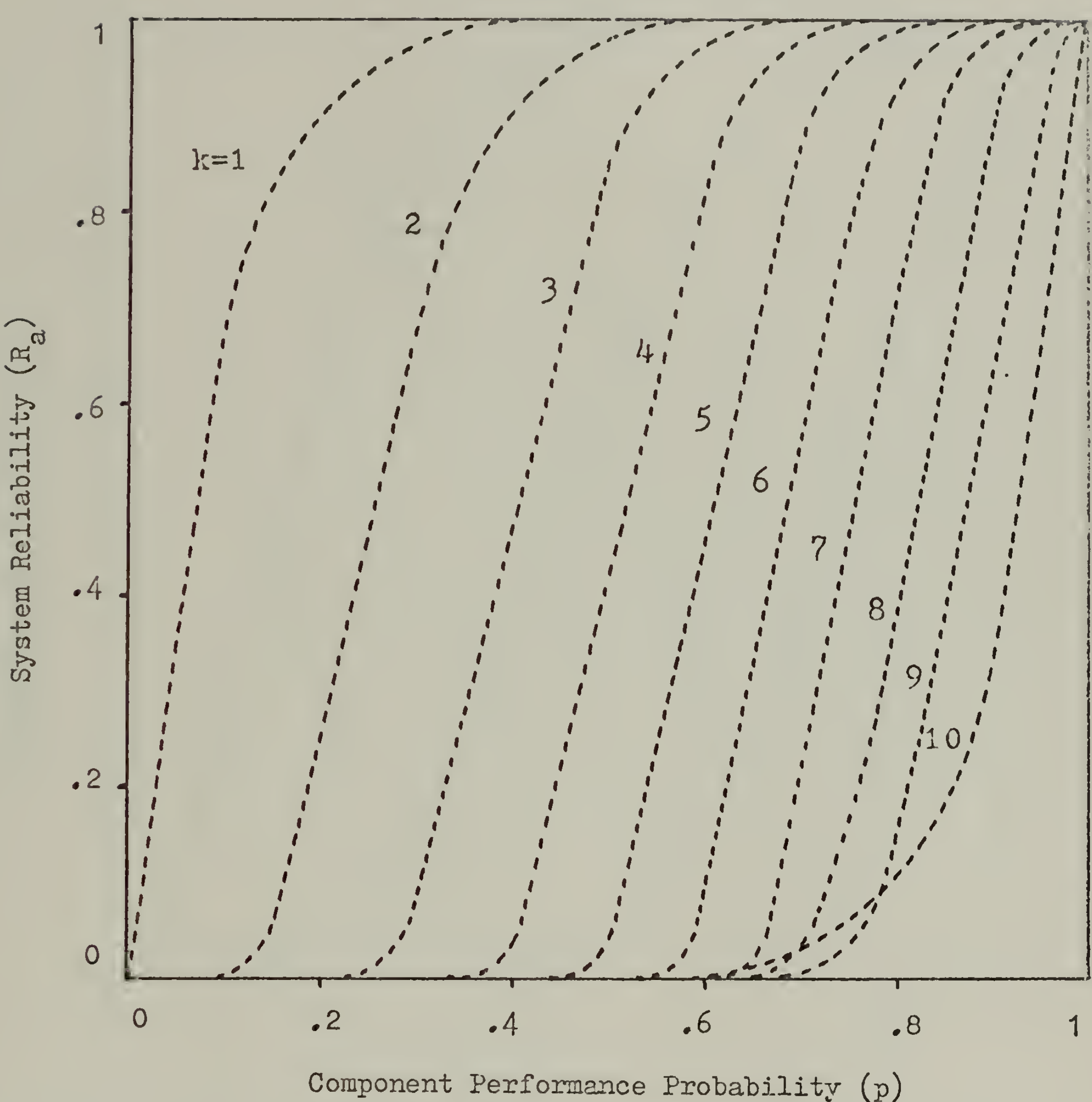

Figure III-C.2

the same in the logical parallel and logical series system. Of interest also is the fact that the logical series system is a better approximation of the true reliability than the minimal cut lower bound when k=8 and 9, and the value of p is approximately .6 to .8. This phenomenon becomes

True Reliabilities and Minimal Cut Approximations

of the k-Out-of-10 Systems

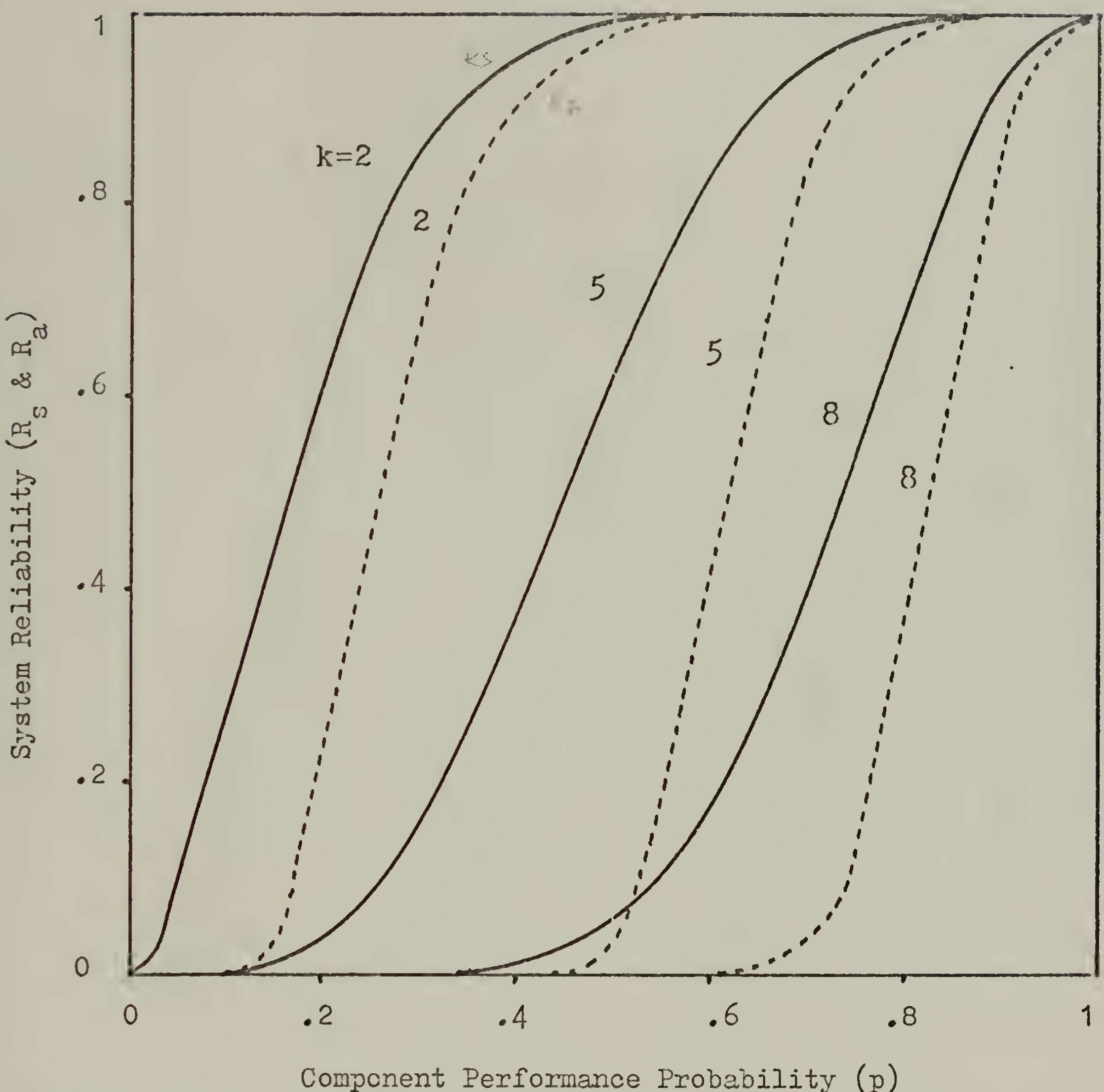

Figure III-C.3

more obvious for the n = 20 systems, and then tends to disappear as n increases further. (See the values listed in the tables of Appendix A.)

Figure III-C.3 establishes the relationship of the minimal cut lower bound to the true system reliability for n=10 and k/n =.2, .5, and .8.

True Reliabilities and Minimal Cut Approximations

of the k-Out-of-20 Systems

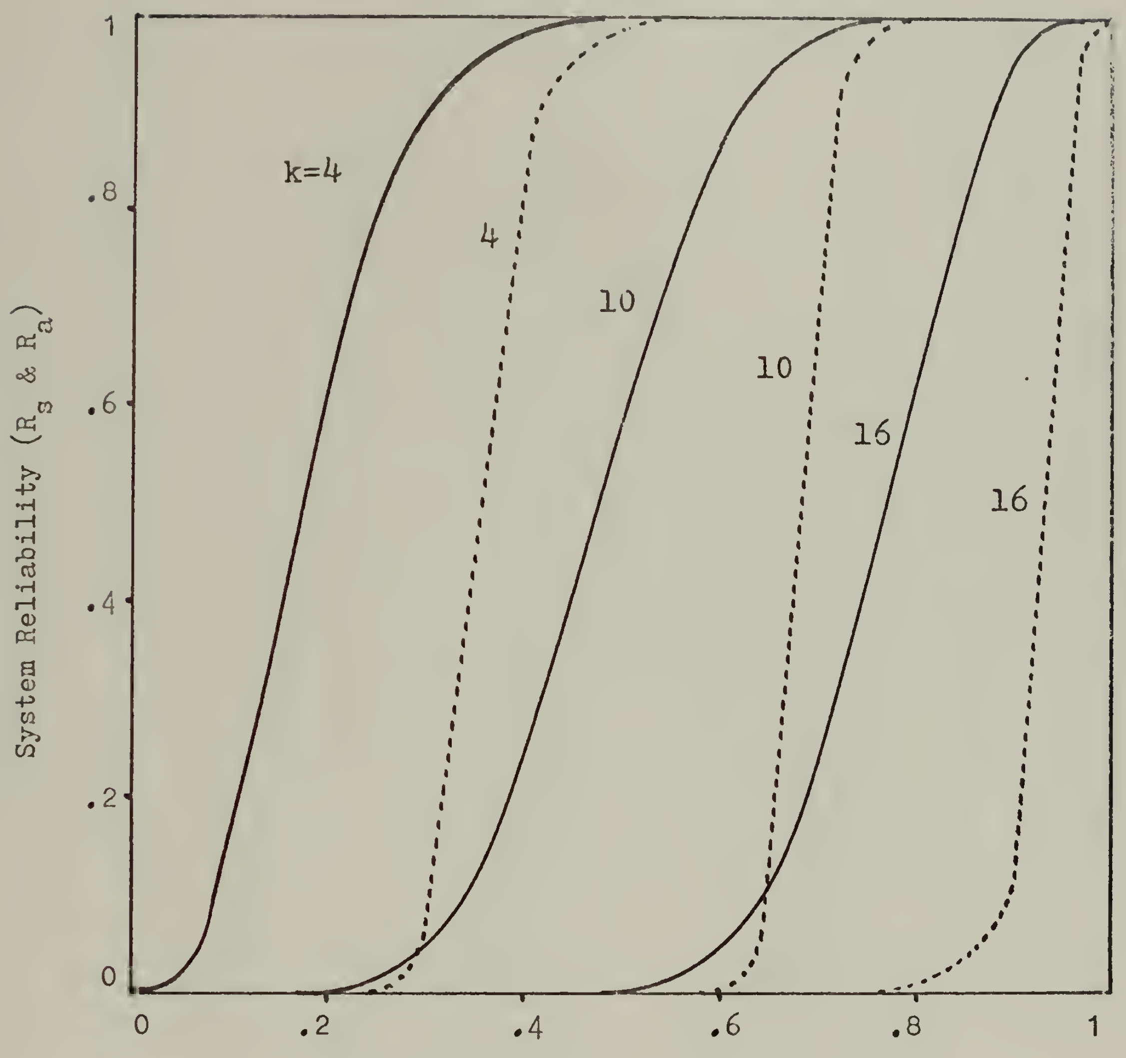

Figure III-C.4

The graph shows that as the k/n ratio increases, higher component performance probability is required to acheive equally good approximations of true reliability. The graph also reveals that as the ratio of k/n increases, the actual deviation of the minimal cut lower bound curve from the true reliability curve increases and then decreases.

True Reliabilities and Minimal Cut Approximations

of the k-Out-of-40 Systems

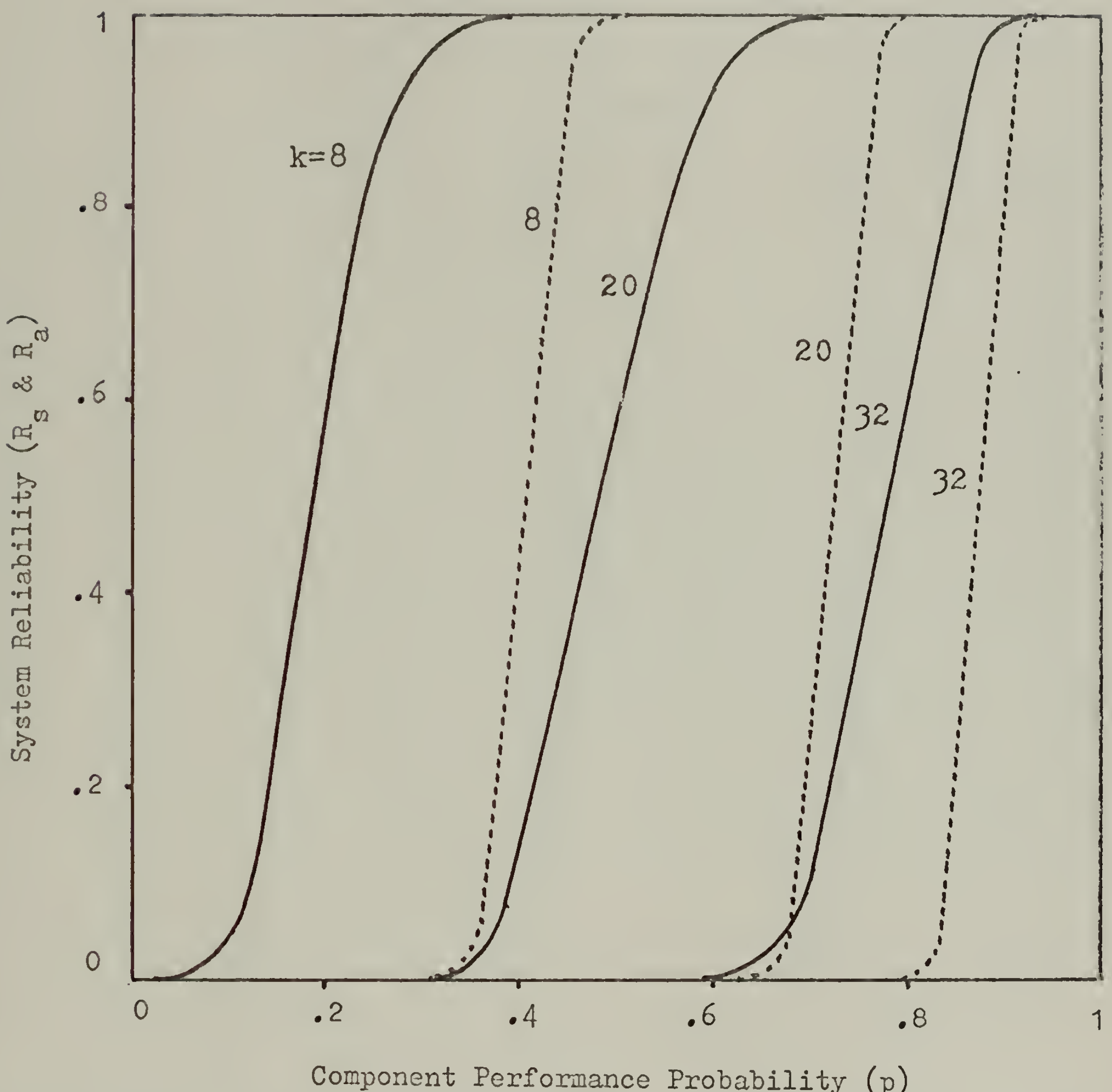

Figure III-C.5

Figures III-C.4, C.5, and C.6 show that as the value of n is increased, the deviation of the two curves for a certain range of p increases. It is also obvious that both types of curves are steepening as the value of n increases. It is known that in the limit as n goes to

True Reliabilities and Minimal Cut Approximations

of the k-Out-of-80 Systems

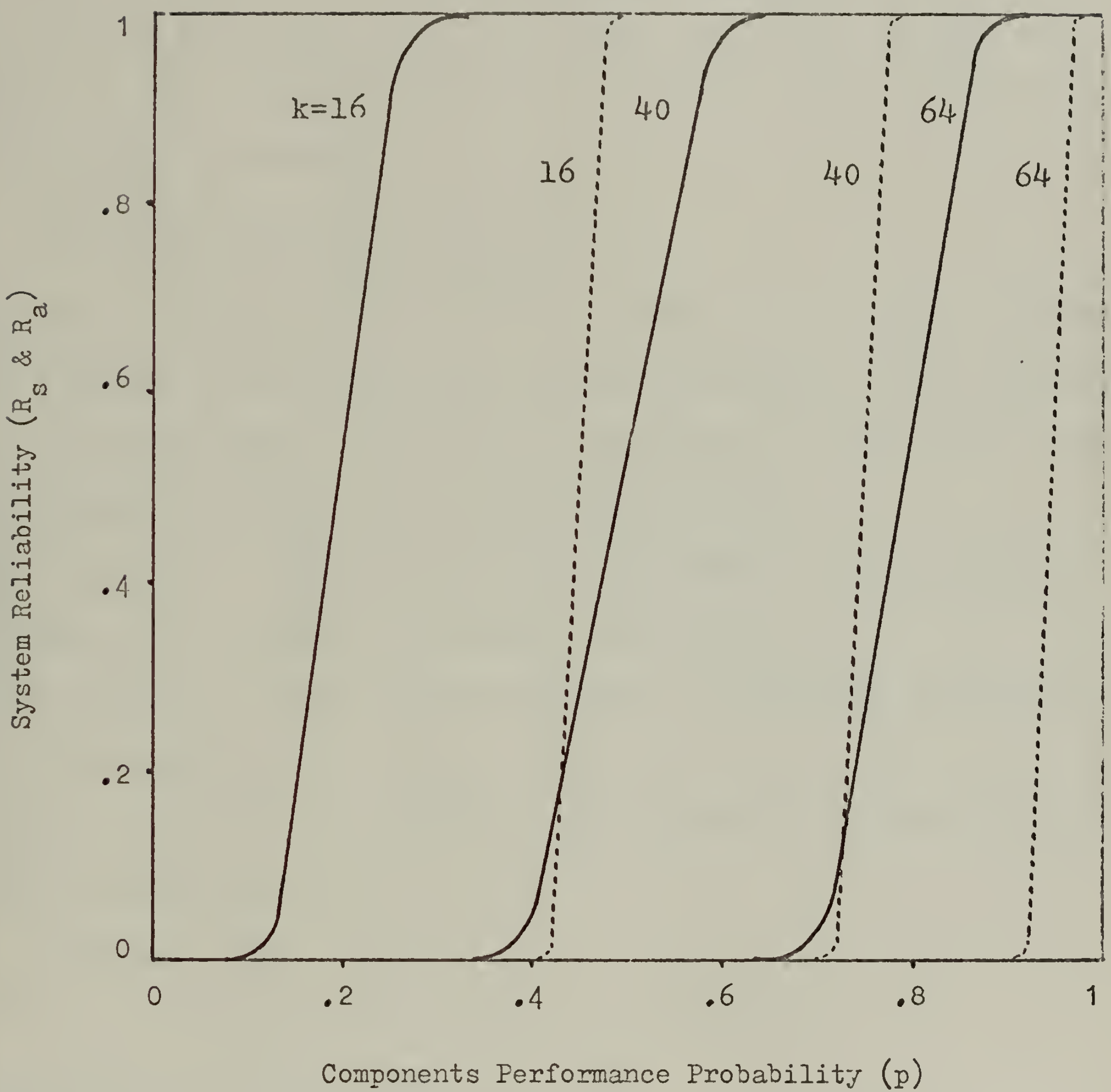

Figure III-C.6

infinity, the reliability of k-out-of-n systems jumps from 0 to 1 at the point where $p = k/n$; and these graphs suggest that as n increases, the minimal cut lower bound may also have some "critical" value of p at which it jumps from 0 to 1. It is clear from the graphs that this critical value of p (p_c = p critical) is also dependent on the k/n ratio.

D. THEORETICAL DISCUSSION

The graphs of section C above indicate that as the number of components in a system increases, the minimal cut lower bound approaches a limit which jumps from 0 to 1 at a critical value, p_c. An analytical proof of this characteristic will now be presented, and a solution for the value of p_c obtained.

The following notation will be used:

$R_a(n,\alpha,p) = \left[1 - (1-p)^{n-k+1}\right]^{\binom{n}{n-k+1}}$; where n is the number of components in a system; α is the ratio of the number of components in a minimal path to the total number of components in the system ($\alpha = k/n$); and p is the performance probability of an individual component.

$L(n) = \binom{n}{n-k+1}$. L(n) denotes the length (number of cut structures in series) of the minimal cut structure representation.

$W(n) = (n-k+1)$. W(n) is the width (number of components in parallel) of a cut structure.

$\delta(n,p) = (1-p)^{n-k+1} = (1-p)^{W(n)}$. $\delta(n,p)$ denotes the probability that a cut structure will fail.

Theorem:
$$\lim_{n\to\infty} R_a(n,\alpha,p) = 1 \qquad \text{if } p \geq p_c$$
$$= 0 \qquad \text{if } p < p_c$$

where $p_c = 1 - \alpha^{\frac{\alpha}{1-\alpha}} + \alpha^{\frac{1}{1-\alpha}}$.

Proof: First, as $n\to\infty$, the reliability of a logical series system ($\alpha = 1$) is 1 at $p = 1$ and 0 for $p < 1$. (See equation (II-C-5).) For the logical parallel system ($\alpha = 1/n$), the reliability as $n \to \infty$ is 0 for $p = 0$ and 1 for $p > 0$. (See equation (II-C-6).) Since these two extreme systems already satisfy the theorem, only the systems with $1/n < \alpha < 1$ will be considered.

Similarly, strictly reliable (p=1) and strictly unreliable (p=0) components obviously produce strictly reliable ($R_a(n,\alpha,p) = 1$) and strictly unreliable ($R_a(n,\alpha,p) = 0$) systems. Again, with the extreme values of p already satisfying the theorem, only components with $0 < p < 1$ will be used.

Note that

$$\lim_{n\to\infty} R_a(n,\alpha,p) = \lim_{n\to\infty} \left[1 - (1-p)^{n-k+1}\right]^{\binom{n}{n-k+1}} \qquad \text{(III-D-1)}$$

$$= \lim_{n\to\infty} [1 - \delta(n,p)]^{L(n)} \quad . \qquad \text{(III-D-2)}$$

Taking logarithms,

$$\lim_{n\to\infty} \ln R_a(n,\alpha,p) = \lim_{n\to\infty} \ln [1 - \delta(n,p)]^{L(n)} \quad .$$

Thus,

$$\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} = \lim_{n\to\infty} L(n) \ln [1 - \delta(n,p)]$$

$$= \lim_{n\to\infty} L(n)\, \delta(n,p) \frac{\ln [1-\delta(n,p)]}{\delta(n,p)} \quad .$$

Since $\lim_{\delta\to 0} \frac{\ln (1-\delta)}{\delta} = -1$, and $\delta(n,p) \to 0$ as $n \to \infty$,

then

$$\lim_{n\to\infty} \ln R_a(n,\alpha,p) = - \lim_{n\to\infty} L(n)\, \delta(n,p) \quad . \qquad \text{(III-D-3)}$$

The expression for L(n) can be simplified. Observe that

$$L(n) = \binom{n}{n-k+1} = \frac{n!}{(n-k+1)!\,(k-1)!} = \frac{k}{(n-k+1)} \frac{n!}{(n-k)!\,k!}$$

$$= \frac{\alpha n}{(1-\alpha)n+1)} \binom{n}{n-k} \sim \frac{\alpha}{(1-\alpha)} \binom{n}{k} \quad ,$$

where $a_n \sim b_n$ means $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Also using Stirling's formula,

$n! \sim n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}}$, observe that $\binom{n}{n-k} = \frac{n!}{(n-k)!\ k!}$

$$\sim \frac{n^{n+\frac{1}{2}} e^{-n} (2\pi)^{\frac{1}{2}}}{(1-\alpha)n^{(1-\alpha)n+\frac{1}{2}} e^{-(1-\alpha)n} (2\pi)^{\frac{1}{2}} (\alpha n)^{\alpha n+\frac{1}{2}} e^{-\alpha n} (2\pi)^{\frac{1}{2}}}$$

$$= \left[\frac{1}{(2\pi)\ n\ (1-\alpha)\ \alpha}\right]^{\frac{1}{2}} \left[\frac{1}{(1-\alpha)^{(1-\alpha)}\ \alpha^{\alpha}}\right]^{n} . \qquad \text{(III-D-4)}$$

Thus, combining equations (III-D-3) and (III-D-4),

$$L(n) \sim \frac{\alpha}{(1-\alpha)} \left[\frac{1}{(2\pi)\ n\ (1-\alpha)\ \alpha}\right]^{\frac{1}{2}} \left[\frac{1}{(1-\alpha)^{(1-\alpha)}\ \alpha^{\alpha}}\right]^{n}$$

$$= \frac{A(\alpha)}{n^{\frac{1}{2}}} \left[\frac{1}{(1-\alpha)^{(1-\alpha)}\ \alpha^{\alpha}}\right]^{n} ,$$

where

$$A(\alpha) = \frac{\alpha^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}}\ (1-\alpha)^{3/2}} .$$

The limit of the function can now be expressed as:

$$\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} = -\lim_{n\to\infty} \frac{A(\alpha)}{n^{\frac{1}{2}}} \left[\frac{1}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}\right]^{n} \delta(n,p)$$

$$= -\lim_{n\to\infty} \frac{A(\alpha)}{n^{\frac{1}{2}}} \left[\frac{1}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}\right]^{n} (1-p)^{(1-\alpha)n+1}$$

$$= -\lim_{n\to\infty} \frac{A'(\alpha)}{n^{\frac{1}{2}}} \left[\frac{1}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}\right]^{n} (1-p)^{(1-\alpha)n} ,$$

where $A'(\alpha) = (1-p)\ A(\alpha)$.

Now,

$$\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} = -\lim_{n\to\infty} \frac{A'(\alpha)}{n^{\frac{1}{2}}} \left[\frac{(1-p)^{(1-\alpha)}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}\right]^n . \qquad \text{(III-D-5)}$$

Let $\theta(\alpha,p) = \dfrac{(1-p)^{(1-\alpha)}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}}$.

If $\theta(\alpha,p) < 1$, then clearly $\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} \to 0.$

If $\theta(\alpha,p) = 1$, then $\dfrac{A'(\alpha)}{n^{\frac{1}{2}}} \to 0$ as $n \to \infty$, and

$\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} \to 0$. If $\theta(\alpha,p) > 1$, then

$$\lim_{n\to\infty} A'(\alpha) \frac{\theta(\alpha,p)^n}{n^{\frac{1}{2}}} = \lim_{n\to\infty} A'(\alpha) \frac{1}{n^{\frac{1}{2}}\left[\frac{1}{\theta(\alpha,p)}\right]^n} \geq \lim_{n\to\infty} A'(\alpha)\frac{1}{n\left[\frac{1}{\theta(\alpha,p)}\right]^n} .$$

Since $\lim_{n\to\infty} n\,\theta(\alpha,p)^{-n} \to 0$, then $\lim_{n\to\infty} A'(\alpha) \dfrac{\theta(\alpha,p)^n}{n^{\frac{1}{2}}} \to \infty$

by the comparison test.

From the preceding arguments, the limiting behavior of $R_a(n,\alpha,p)$ depends on whether $\theta(\alpha,p) \leq 1$ or $\theta(\alpha,p) > 1$. The inequality,

$$\theta(\alpha,p) = \frac{(1-p)^{(1-\alpha)}}{(1-\alpha)^{(1-\alpha)}\alpha^{\alpha}} \leq 1 ,$$

is equivalent to

$$(1-p)^{(1-\alpha)} \leq (1-\alpha)^{(1-\alpha)}\alpha^{\alpha} ,$$

or to

$$p \geq 1 - (1-\alpha)\,\alpha^{\frac{\alpha}{1-\alpha}} = 1-\alpha^{\frac{\alpha}{1-\alpha}} + \alpha^{\frac{1}{1-\alpha}} = p_c . \qquad \text{(III-D-6)}$$

The proof can now be summarized in the following statement:

If $p < p_c$, then $\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} = -\infty$ and $\lim_{n\to\infty} R_a(n,\alpha,p) = 0$. If $p \geq p_c$, then $\lim_{n\to\infty} \{\ln R_a(n,\alpha,p)\} = 0$ and $\lim_{n\to\infty} R_a(n,\alpha,p) = 1$.

Q.E.D.

Example: For a k-out-of-n type system, select a k/n ratio of .8. The critical value of p is then:

$$p_c = 1 - .8^{.8/.2} + .8^{1/.2}$$

$$= 1 - .41 + .328$$

$$= .918 \quad .$$

In the limit, as $n \to \infty$, the minimal cut lower bound is 0 if $p < .918$ and is 1 if $p \geq .918$.

The table below gives the critical values of p for some selected k/n ratios.

k/n	.1	.2	.3	.4	.5	.6	.7	.8	.9
p_c	.303	.465	.582	.674	.750	.814	.869	.918	.961

As stated earlier, it is already known that $R_s(n,\alpha,p)$, as n goes to infinity, places all of its probability mass at k/n. Figure III-D.1 is, then, a graph of the critical values of p for both the true system reliability and the minimal cut lower bound. From this figure it can be seen that the greatest deviation occurs at a k/n ratio of .3. The area between the two curves represents the range of values of p for which the minimal cut lower bound is not a good approximation. This graph, of course, is valid only in the limit as n goes to infinity.

Critical Values of p for the k-Out-of-n Systems

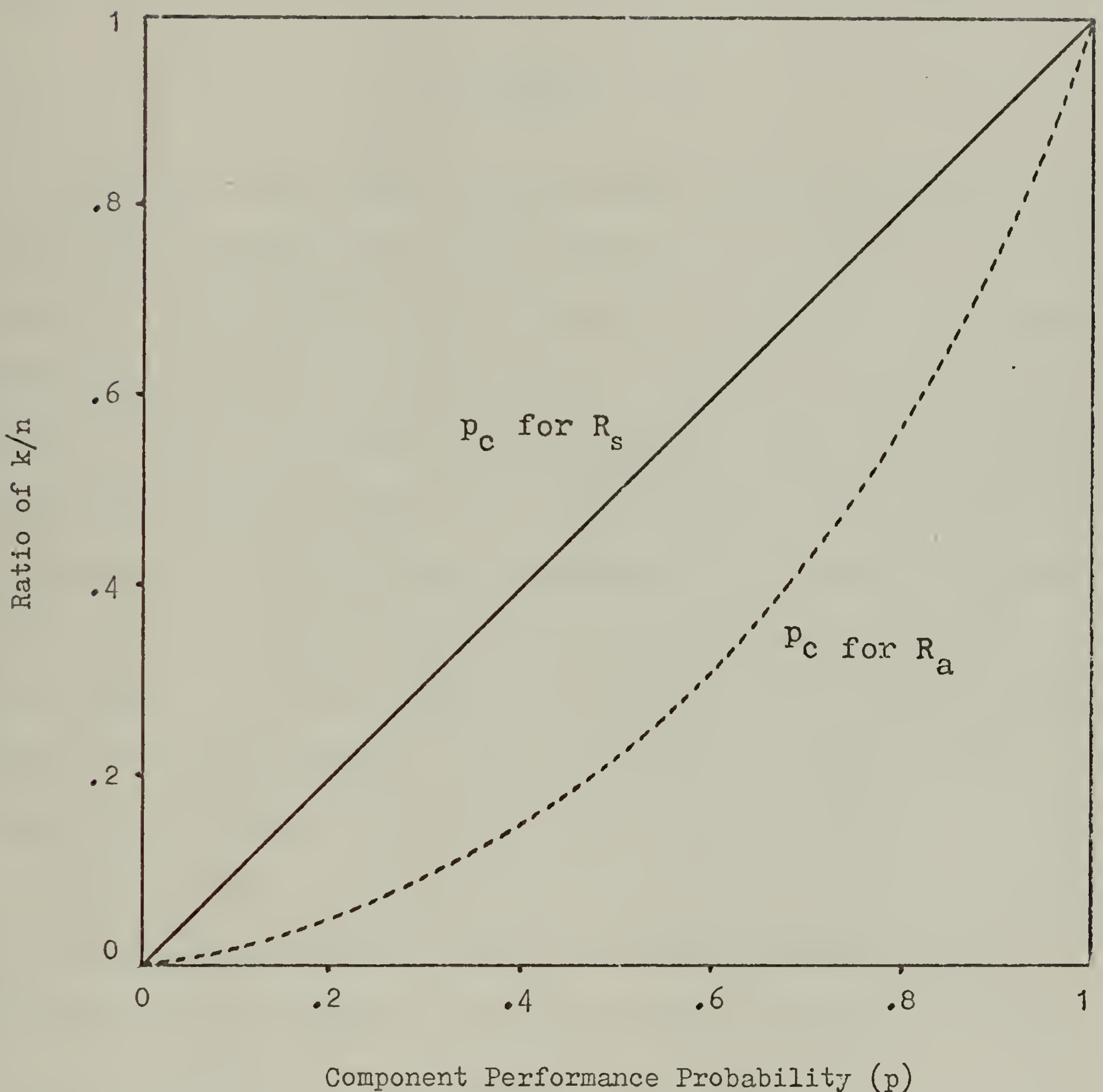

Figure III-D.1

A review of figure III-C.6 reveals that at n = 80, the minimal cut lower bound is already approaching its limiting form. This suggests that the lower bound converges to its limit much faster than the true system reliability and offers one explanation as to why the logical series

system is sometimes a better approximation than the minimal cut lower bound. It also suggests that the graph of p_c for the lower bound in figure III-D.1 may have application to some rather modest size systems.

IV. SUMMARY

It is important to state once again that because the reliability of the k-out-of-n system can be found directly and accurately, the minimal cut lower bound for this type of system has no real value. The k-out-of-n system does, however, allow a comparison between true relaibility and the lower bound to be made; and this comparison offers the opportunity to make general statements about the characteristics of the approximation.

Increasing the size (number of components) and complexity (number of cut sets) of a system causes a rapid deterioration of the minimal cut lower bound. Of the two, complexity appears to have a more pronounced effect on the approximation. The deterioration does have a limit, though; and this limit seems to be reached rather quickly.

In spite of the deterioration of the lower bound noted in this paper, the minimal cut lower bound remains a valid approximation. This phenomenon can be restated in the following way: If the lower bound gives an approximation to the system's reliability that is acceptable, then it can be used with confidence. If the approximation is not acceptable, then caution should be exercised before rejecting or redesigning the system. The system may already be very reliable.

APPENDIX A

Tables of Reliabilities and Approximations

n = the total number of components in the system.

k = the number of components in a minimal path.

p = the probability the component functions properly.

$$R_s = \sum_{k'=k}^{n} \binom{n}{k'} p^{k'} (1-p)^{n-k'}$$.[1]

$$R_a = \left[1 - (1-p)^{n-k+1}\right]^{\binom{n}{n-k+1}}$$.[2]

---

[1]Figures obtained from Tables of the Cumulative Binomial Probability Distribution, Harvard University Press, 1955; and rounded off to four significant decimals.

[2]Figures computed on an IBM-360, U.S. Naval Postgraduate School, and rounded off to four significant decimals.

k-OUT-OF-10 SYSTEMS

k	p	R_s	R_a	k	p	R_s	R_a
1	.1	.6513	.6513	6	.1	.0001	.0000
	.2	.8926	.8926		.2	.0064	.0000
	.3	.9718	.9718		.3	.0473	.0000
	.4	.9940	.9940		.4	.1662	.0000
	.5	.9990	.9990		.5	.3770	.0003
	.6	.9999	.9999		.6	.6331	.0747
	.7	1.0000	1.0000		.7	.8497	.5417
	.8	1.0000	1.0000		.8	.9672	.9225
	.9	1.0000	1.0000		.9	.9984	.9975
2	.1	.2639	.0074	7	.1	.0000	.0000
	.2	.6242	.2367		.2	.0009	.0000
	.3	.8507	.6624		.3	.0106	.0000
	.4	.9536	.9037		.4	.0548	.0000
	.5	.9893	.9806		.5	.1719	.0000
	.6	.9983	.9974		.6	.3823	.0043
	.7	.9999	.9998		.7	.6496	.1812
	.8	1.0000	.9999		.8	.8791	.7144
	.9	1.0000	1.0000		.9	.9872	.9792
3	.1	.0702	.0000	8	.1	.0000	.0000
	.2	.3222	.0002		.2	.0001	.0000
	.3	.6172	.0691		.3	.0016	.0000
	.4	.8327	.4666		.4	.0123	.0000
	.5	.9453	.8385		.5	.0547	.0000
	.6	.9897	.9709		.6	.1673	.0000
	.7	.9984	.9970		.7	.3828	.0375
	.8	.9999	.9998		.8	.6778	.3874
	.9	1.0000	.9999		.9	.9298	.8869
4	.1	.0128	.0000	9	.1	.0000	.0000
	.2	.1209	.0000		.2	.0000	.0000
	.3	.3504	.0000		.3	.0001	.0000
	.4	.6177	.0331		.4	.0017	.0000
	.5	.8281	.3902		.5	.0107	.0000
	.6	.9452	.8214		.6	.0464	.0004
	.7	.9894	.9741		.7	.1493	.0143
	.8	.9991	.9985		.8	.3758	.1593
	.9	1.0000	.9999		.9	.8361	.6362
5	.1	.0016	.0000	10	.1	.0000	.0000
	.2	.0328	.0000		.2	.0000	.0000
	.3	.1503	.0000		.3	.0000	.0000
	.4	.3669	.0000		.4	.0001	.0001
	.5	.6230	.0366		.5	.0010	.0010
	.6	.8338	.4223		.6	.0060	.0060
	.7	.9527	.8580		.7	.0282	.0282
	.8	.9936	.9866		.8	.1074	.1074
	.9	.9999	.9998		.9	.3487	.3487

k-OUT-OF-20 SYSTEMS

k	p	R_s	R_a	k	p	R_s	R_a
2	.1	.6083	.0549	12	.1	.0000	.0000
	.2	.9308	.7480		.2	.0001	.0000
	.3	.9924	.9774		.3	.0051	.0000
	.4	.9995	.9988		.4	.0565	.0000
	.5	1.0000	.9999		.5	.2517	.0000
	.6	1.0000	1.0000		.6	.5917	.0000
	.7	1.0000	1.0000		.7	.8867	.0366
	.8	1.0000	1.0000		.8	.9900	.9176
	.9	1.0000	1.0000		.9	.9999	.9998
4	.1	.1330	.0000	14	.1	.0000	.0000
	.2	.5886	.0000		.2	.0000	.0000
	.3	.8929	.0703		.3	.0003	.0000
	.4	.9840	.8245		.4	.0065	.0000
	.5	.9987	.9913		.5	.0577	.0000
	.6	1.0000	.9998		.6	.2500	.0000
	.7	1.0000	.9999		.7	.6080	.0000
	.8	1.0000	1.0000		.8	.9133	.3707
	.9	1.0000	1.0000		.9	.9976	.9923
6	.1	.0013	.0000	16	.1	.0000	.0000
	.2	.1958	.0000		.2	.0000	.0000
	.3	.5836	.0000		.3	.0000	.0000
	.4	.8744	.0007		.4	.0003	.0000
	.5	.9793	.6230		.5	.0059	.0000
	.6	.9984	.9835		.6	.0510	.0000
	.7	1.0000	.9998		.7	.2375	.0000
	.8	1.0000	.9999		.8	.6296	.0070
	.9	1.0000	1.0000		.9	.9563	.8564
8	.1	.0004	.0000	18	.1	.0000	.0000
	.2	.0321	.0000		.2	.0000	.0000
	.3	.2277	.0000		.3	.0000	.0000
	.4	.5841	.0000		.4	.0000	.0000
	.5	.8684	.0001		.5	.0002	.0000
	.6	.9790	.5944		.6	.0036	.0000
	.7	.9987	.9877		.7	.0355	.0000
	.8	1.0000	.9999		.8	.2061	.0001
	.9	1.0000	1.0000		.9	.6769	.3196
10	.1	.0000	.0000	20	.1	.0000	.0000
	.2	.0026	.0000		.2	.0000	.0000
	.3	.0480	.0000		.3	.0000	.0000
	.4	.2447	.0000		.4	.0000	.0000
	.5	.5881	.0000		.5	.0000	.0000
	.6	.8725	.0009		.6	.0000	.0000
	.7	.9829	.7426		.7	.0008	.0008
	.8	.9994	.9966		.8	.0115	.0115
	.9	1.0000	.9999		.9	.1216	.1216

k-OUT-OF-40 SYSTEMS

k	p	R_s	R_a	k	p	R_s	R_a
4	.1	.5769	.0000	24	.1	.0000	.0000
	.2	.9715	.0769		.2	.0000	.0000
	.3	.9994	.9818		.3	.0001	.0000
	.4	1.0000	.9999		.4	.0083	.0000
	.5	1.0000	1.0000		.5	.1341	.0000
	.6	1.0000	1.0000		.6	.5681	.0000
	.7	1.0000	1.0000		.7	.9367	.0000
	.8	1.0000	1.0000		.8	.9991	.8902
	.9	1.0000	1.0000		.9	1.0000	1.0000
8	.1	.0419	.0000	28	.1	.0000	.0000
	.2	.5628	.0000		.2	.0000	.0000
	.3	.9447	.0000		.3	.0000	.0000
	.4	.9979	.4105		.4	.0001	.0000
	.5	.9999	.9978		.5	.0083	.0000
	.6	1.0000	.9999		.6	.1285	.0000
	.7	1.0000	1.0000		.7	.5772	.0000
	.8	1.0000	1.0000		.8	.9568	.0000
	.9	1.0000	1.0000		.9	.9999	.9988
12	.1	.0004	.0000	32	.1	.0000	.0000
	.2	.1182	.0000		.2	.0000	.0000
	.3	.5594	.0000		.3	.0000	.0000
	.4	.9290	.0000		.4	.0000	.0000
	.5	.9968	.0135		.5	.0001	.0000
	.6	.9999	.9933		.6	.0061	.0000
	.7	1.0000	.9999		.7	.1111	.0000
	.8	1.0000	1.0000		.8	.5931	.0000
	.9	1.0000	1.0000		.9	.9845	.7607
16	.1	.0000	.0000	36	.1	.0000	.0000
	.2	.0052	.0000		.2	.0000	.0000
	.3	.1151	.0000		.3	.0000	.0000
	.4	.5598	.0000		.4	.0000	.0000
	.5	.9230	.0000		.5	.0000	.0000
	.6	.9937	.0108		.6	.0001	.0000
	.7	1.0000	.9966		.7	.0026	.0000
	.8	1.0000	1.0000		.8	.0759	.0000
	.9	1.0000	1.0000		.9	.6290	.0014
20	.1	.0000	.0000	40	.1	.0000	.0000
	.2	.0000	.0000		.2	.0000	.0000
	.3	.0062	.0000		.3	.0000	.0000
	.4	.1298	.0000		.4	.0000	.0000
	.5	.5627	.0000		.5	.0000	.0000
	.6	.9257	.0000		.6	.0000	.0000
	.7	.9976	.2533		.7	.0000	.0000
	.8	1.0000	.9997		.8	.0002	.0002
	.9	1.0000	1.0000		.9	.0148	.0148

k-OUT-OF-80 SYSTEMS

k	p	R_s	R_a
16	.1	.0053	.0000
	.2	.5445	.0000
	.3	.9839	.0000
	.4	.9999	.0000
	.5	1.0000	1.0000
	.6	1.0000	1.0000
	.7	1.0000	1.0000
	.8	1.0000	1.0000
	.9	1.0000	1.0000
40	.1	.0000	.0000
	.2	.0000	.0000
	.3	.0001	.0000
	.4	.0445	.0000
	.5	.5445	.0000
	.6	.9729	.0000
	.7	.9999	.0000
	.8	1.0000	1.0000
	.9	1.0000	1.0000
64	.1	.0000	.0000
	.2	.0000	.0000
	.3	.0000	.0000
	.4	.0000	.0000
	.5	.0000	.0000
	.6	.0002	.0000
	.7	.0302	.0000
	.8	.5664	.0000
	.9	.9979	.0000

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The minimal cut lower bound for k-out-of-n systems is computed and compared with the true reliability of these systems. The size of the system, n, is increased; and selected degrees of system complexity, k/n, are studied. The resulting graphs of system reliability versus component reliability indicate that both size and complexity cause a deterioration of the approximation, but they also indicate that there is a limit to this deterioration. The minimal cut lower bound is then examined, theoretically, as the size of the system increases to infinity; and the limits of deterioration are obtained.

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